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S and P wave $\pi\pi$ scattering amplitude by extrapolation

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The S and P partial wave amplitudes of pion-pion scattering are constructed using analyticity, unitarity, crossing symmetry, a CDD pole and the ideas of optimal convergence through conformal mapping. On extrapolation to the physical region they yield phase shifts and inelasticities which are in good agreement with the presently available experimental data upto the centre of mass energy of 1.4 GeV.

1. Introduction

The study of pion-pion scattering has gained considerable importance with the availability of experimental data for quite high energies One hopes to have a clear understanding of strong interactions if the pion-pion scattering results are understood in a consistent theory Various attempts have been made to construct models and explain the experimental data in the low energy region (<0.9 GeV). Using current commutation relation and the partially conserved axial vector current (PCAC) Weinberg (1966) has obtained expressions for the pion-pion scattering lengths Brown & Goble (1968) imposed elastic unitarity on it and tried to extrapolate this amplitude above threshold by using a relativistic version of the effective range approximation Their amplitude cannot be shown to follow from a crossing symmetric theory which the original current algebra amplitude explicitly had Krinsky (1970) tried to correct this defect by using Roskies (1969) sum rules In the mean time Wanders & Piguet (1968) had tried to construct neutral theory models satisfying Martin's (1967) inequalities and investigate the physical content of the corssing constraints Their models were based on the parametrization of the inverse K-matrix element $\phi(s) = \frac{1}{2}k \cot \delta(s)$ and was characterised by a number of poles of $\phi(s)$. On the other hand Kang *et al* (1971) carried out a complete K-matrix unitarization of the Veneziano amplitudes In their case crossing symmetry was enforced by calculating Roskies' relations and Martin's inequalities. Analyticity was also evident in the sense that the partial wave amplitudes possess a nearby left hand cut in addition to the right hand elastic cut. However, the $l = 0$, $I = 0$ phase shifts predicted by them up to 0.9 GeV lie in between the so called *up-up* and *down-down* (Malamud & Schlein 1967) solutions Basdevant & Lee (1970) unitarized the current algebra amplitudes based on the σ model of Gell-Mann & Levy (1960) and the theory of Pade approximants. The amplitude so constructed satisfy the current algebra low energy theorems (Adler 1965) and are partially crossing symmetric

Auberson *et al* (1968) proposed a model for *s*-wave which satisfies Martin's inequalities. Kang & Lee (1971) used the N/D method and approximated the left hand singularities by a few poles The parameters were determined by using the Roskies' relations and current algebra conditions They have demonstrated there that the low energy theorems (Adler 1965) of current algebra and the existence of the ρ and σ mesons as two pion bound states can be incorporated in a set of unitary and crossing symmetric amplitudes But Tryon (1972) has shown that the Kang and Lee model violates analyticity and/or crossing summetry and Regge sum rules for *S*- and *P*- wave scattering lengths, mainly because the number of poles used to simulate left hand cuts are too small Recently Bonnier & Gauron (1972) used the correct mass and width of the rho meson and tried to construct *S*, *P*, and *D*-waves so as to be in agreement with the rigorous results deduced from analyticity, unitarity and crossing symmetry Their results give a well defined σ meson with a mass, $m_\sigma \simeq 0.7$ GeV and a width, $\Gamma \simeq 0.26$ GeV

All the above efforts have been confined to energy less than 0.9 GeV More over, now that the *up-up* and *down-down* ambiguity is resolved (Particle Data Group 1972) and experimental data are available up to 1.4 GeV most of the above models will need reexamination * Instead of recalculating we decided to exploit the unphysical region constraints along with unitarity and analyticity and construct the *S* and *P* partial wave amplitudes To this end we use the technique of Optimized Polynomial Expansion due to Cutkosky & Deo (1968) tailored to integrate Roskies' relations most efficiently and still offer an efficient way of extrapolation to the physical region. The method, in actuality, consists of mapping the symmetrical cut planes into the interior of an ellipse and of constructing the partial wave amplitudes as polynomials in the mapped variable Besides the Roskies' relations, the inequalities due to Martin & Pennington (1970, 1971) are used as inputs to define a starting set of parameters. This also ensures that the partial wave amplitudes constructed by us have manifest-analyticity and satisfy the constraints enforced by unitarity and crossing and are thus derivable from a set of crossing symmetric scattering amplitudes. We, further, take note of the fact that when we approach $s = 0$ from the negative side, the partial wave amplitudes have a $\sqrt{s}$ type singularity and their imaginary parts go as $(-s)^{3/2}$

*After this work was completed we have learnt that Basdevant *et al* (1973) have used conformal mapping variable to parameterise the $\pi-\pi$ scattering data Our motivation is, however, entirely different.

It has been shown by Atkinson (1968) that analyticity, unitarity and crossing alone do not determine the amplitudes uniquely. Some dynamical input like PCAC or CDD poles or similar concepts are needed Atkinson *et al* (1972) have also shown that the Legendre expansion of the scattering amplitude may not in general converge if there is a CDD pole in partial waves of angular momentum $l > 0$ Thus for $l = 0$, one could have CDD poles both in $I = 0$ and $I = 2$ waves Since $I = 2$, $l = 0$ wave is exotic, it is unlikely to contain a CDD pole which could lead to resonances/antiresonances Fortunately we find that we need only one CDD pole for $l = 0$, $I = 0$ wave. Moreover, by innumerable laborious searches we observed that it is not possible to satisfy all the constraints without this CDD pole. It is not possible to prove uniqueness of our extrapolation scheme We claim, however, that the inputs are only mildly model dependent We neither use the shape and position of the rho resonance, the current algebra constraints, or the dual model results of Veneziano (1968) nor the experimental values for phase shifts and inelasticities

In a continuing paralled work (Deo & Mohapatra 1975), we have studied the phenomenology of $\pi\pi$ scattering by conformal mapping method In phenomenological study all the experimental data are analysed The data are fitted by equations involving polynomials and the coefficients of expansion are varied to give the best fit or least χ^2. The purpose of this paper is not to use any experimental data, but to predict them This is possible by extrapolating the curves of the unphysical region to the physical region Since the conformal mapping method is a powerful tool for extrapolation, we have hoped to *predict* in this paper the phase shifts and inelasticities as they are given by the experiment Our predictions for the phase shifts match well with the experiment However, our prediction for η_0^0 agrees only qualitatively with the experimental data But in the absence of any physical constraint to keep the values of $\eta_0^0 = 1$, upto the $K\bar{K}$ threshold any natural extrapolation will perhaps give a value of $\eta_0^0 < 1$, because in the absence of inelasticity for $4m_\pi^2 \leqslant S \leqslant 4m^2_K$ the amplitude will identically vanish (Martin 1969)

The paper is organized as follows . In section 2 we review the elastic unitarity conditions and crossing conditions on the partial waves as well as well as their analyticity in the complex s-plane In section 3 partial wave amplitudes are constructed In section 4 details of our calculations are given In section 5 we present numerical results Some coments and discussion of results are made in section 6 in our concluding remarks

2 Preliminaries

Unitarity ·

We consider a typical process :

$$\pi^\alpha(k_1)+\pi^\beta(k_2)\rightarrow \pi^\gamma(k_3)+\pi^\delta(k_4), \qquad \dots (1)$$

where $\alpha, \beta, \gamma, \delta$ are the isospin indices. The scattering amplitude F^I, for the isospin state I is expanded into the partial waves as,

$$F^I(s, t) = \sum_{l=0}^{\infty} (2l+1) f_l^I(s) P_l(\cos\theta), \qquad \ldots \quad (2)$$

where s and t are the usual Mandelstam variables and $\cos\theta = 1+2t/(s-1)$ We choose our units in such a way that $s = k^2+1$ Then the elastic unitarity condition for partial wave amplitudes is given by

$$Im(f_l^I(s))^{-1} = -\rho(s), \qquad \ldots \quad (3)$$

$$\rho(s) = \frac{k}{2\sqrt{s}} \qquad \ldots \quad (4)$$

This implies the familiar partial wave formulation (Frye & Warnock 1963) in terms of the phase shifts $\delta_l^I(s)$ as,

$$f_l^I(s) = \frac{k^{2l}[\eta_l^I \exp(2i\delta_l^I)-1]}{2i\phi_l(s)} \qquad \ldots \quad (5)$$

where

$$\phi_l(s) = \frac{k^{2l+1}}{2\sqrt{s}}, \qquad \ldots \quad (6)$$

and η_l^I is the inelasticity which is given by,

$$\eta_l^I = \left| 1+\frac{2i\phi_l(s) f_l^I(s)}{k^{2l}} \right|, \qquad \ldots \quad (7)$$

so that $\eta_l^I < 1$ for inelastic processes, $\eta_l^I = 1$ for elastic processes. We find that $\eta_l^I \leqslant 1$ is a very strong constraint for extrapolation into the physical region

Crossing :

The most general form of pion-pion amplitude is,

$$T^{\alpha\beta\gamma\delta}_{(k_1k_2k_3k_4)} = \delta_{\alpha\beta}\delta_{\gamma\delta}A(s, t, u)+\delta_{\alpha\gamma}\delta_{\beta\delta}B(s, t, u)+\delta_{\alpha\delta}\delta_{\beta\gamma}C(s, t, u) \qquad \ldots \quad (8)$$

Crossing symmetry leads at once to the relations,

$$A(s, t, u) = A(s, t, u),$$

$$B(s, t, u) = A(t, u, s)$$

$$C(s, t, u) = A(u, s, t). \qquad \ldots \quad (9)$$

There are many relations which follow by using the crossing conditions on the partial wave amplitudes.

(i)By a systematic application of the ideas of Balchandran & Nuyts (1968), Roskies (1969), Deo & Patnaik (1972) expressed the crossing conditions in the form of equalities involving integrals over S and P partial wave amplitudes in the unphysical region $0 \leqslant s \leqslant 1$. They are,

$$\int_0^1 (1-s)f_0^0(s)ds = \int_0^1 \frac{5}{2}(1-s)f_0^2(s)ds, \quad \ldots \quad \text{(R-1)}$$

$$\int_0^1 (1-s)(3s-1)f_0^0(s)ds = -2\int_0^1 (1-s)(3s-1)f_0^2(s)ds, \quad \ldots \quad \text{(R-2)}$$

$$\int_0^1 s(1-s)(2f_0^0(s)-5f_0^2(s))ds = -3\int_0^1 (1-s)^2 f_1^1(s)ds, \quad \ldots \quad \text{(R-3)}$$

$$\int_0^1 s(1-s)^2(2f_0^0(s)-5f_0^2(s))ds = -3\int_0^1 s(1-s)^2 f_1^1(s)ds, \quad \ldots \quad \text{(R-4)}$$

$$\int_0^1 s(1-s)^3(2f_0^0(s)-5f_0^2(s)ds = -3\int_0^1 s(1-s)^2(3s-1)f_1^1(s)ds. \quad \ldots \quad \text{(R-5)}$$

The importance of these relations lies in the fact that if a set of partial wave amplitudes are found which satisfy them, there exists a fully crossing symmetric $\pi-\pi$ amplitude from which this set can be gotten by partial wave projection.

(ii) The second most useful relations are due to Martin (1967), based on the crossing symmety and the positivity of the absorptive part of the total amplitude If we denote $f_0(s)$ as,

$$f_0(s) = (f_0^0(s)+2f_0^2(s))/3 \quad \ldots \quad (10)$$

then Martin's inequalities are

$$f_0(0.80125) > f_0(0.05335) \quad \ldots \quad \text{(M-1)}$$
$$f_0(0.05335) > f_0(0\ 746575) \quad \ldots \quad \text{(M-2)}$$
$$f_0(0) \geqslant f_0(0.7875) \quad \ldots \quad \text{(M-3)}$$
$$f_0(s) < f_0(1) \quad 0 \leqslant s < 1 \quad \ldots \quad \text{(M-4)}$$

$$\frac{d}{ds}(f_0(s) < 0 \quad ; \quad 0 \leqslant s \leqslant 0.3225 \quad \ldots \quad \text{(M-5)}$$

$$\frac{d}{ds}(f_0(s) > 0; \quad ; \quad s \geqslant 0\ 44 \quad \ldots \quad \text{(M-6)}$$

The thirteen inequalities, due to Pennington (1970, 1971) which are also used by our analysis, are listed in the appendix for sake of completeness and for reporting our values for the inequalities.

Analyticity :

The s-channel invariant amplitude can be expanded as,

$$A(s, t, u) = \sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(\cos\theta) \quad \ldots \quad (11)$$

which gives,

$$f_l(s) = \frac{1}{2} \int_{-1}^{1} A(s, t', u') P_l(\cos \theta') d \cos \theta'. \tag{12}$$

Now there are two possible sources of singularity in $f_l(s)$ as defined in eq (13). One gives the right hand cut, the elastic cut from $s = 1$ to $s = \infty$ and the buried inelastic cut from $s = 4$ to $s = \infty$

The left hand cut starts from $s = 0$ and runs to $-\infty$. We wish to study the nature of this cut in greater detail not because the singularity is of the type $(-s)^{3/2}$ but because to find the start of the equivalent inelastic cut for $s < 0$. We write a fixed s dispersion relation for $A(s, t', u')$ in the crossed channels,

$$A(s, t', u') = \frac{1}{\pi} \int_1^{\infty} \left[\frac{ImA_t(s, t_1)}{t_1 - t'} dt_1 + \frac{ImA_u(s, u_1)}{u_1 - u'} du_1 \right]$$

$$+ \frac{1}{\pi} \int_4^{\infty} \left[\frac{ImA_t(s, t_1)}{t_1 - t'} dt_1 + \frac{ImA_u(s, u_1)}{u_1 - u'} du_1 \right] \tag{13}$$

The lower limits of the 1st and 2nd integrals of eq (13) corresponds to the elastic and inelastic thresholds, respectively, of the crossed channels. Inserting eq (13) in eq. (12) one gets,

$$f_l(s) = \frac{1}{q^2 \sqrt{s}} \left[\int_1^{\infty} (D_t dt_1 + D_u du_1) + \int_4^{\infty} (D_t dt_1 + D_u du_1), \right. \tag{14}$$

where
$$D_t = ImA_t(s, t_1) Q_l \left(1 + \frac{2t_1}{q^2} \right),$$

and
$$D_u = ImA_u(s, u_1)\, Q_l \left(-1 - \frac{2u_1}{q^2} \right) \tag{15}$$

But $Q_l(\xi)$ has got a cut for $\xi = -1$ to $\xi = +1$. Thus the first integral gives rise to a cut for $-\infty \leqslant q^2 \leqslant -1$ and the second integral gives rise to a cut for $-\infty \leqslant q^2 \leqslant -4$. Hence in the s-plane $f_l(s)$ will have cut from 0 to $-\infty$ having the multiparticle cut from -3 to $-\infty$. Utilising eq (12) one obtains an expansion of the partial waves around $s = 0$. This has the form (Roskies 1970)

$$f_0^0(s) = f_0^0(0) + s f_0^{0\prime}(0) - i \frac{2}{3} \left[\frac{1}{3} |f_0^0(1)|^2 + \frac{5}{3} |f_0^2(1)|^2 \right] (-s)^{3/2} \qquad \ldots \tag{17}$$

$$f_0^2(s) = f_0^2(0) + s f_0^{2\prime}(0) - i \frac{2}{3} \left[\frac{1}{3} |f_0^0(1)|^2 + \frac{1}{6} |f_0^2(1)|^2 \right] (-s)^{3/2}, \qquad \ldots \tag{18}$$

$$f_1^1(s) = f_1^1(0) + s f_1^{1\prime}(0) + i \frac{2}{3} \left[\frac{1}{3} |f_0^0(1)|^2 - \frac{5}{6} |f_0^2(1)|^2 \right] (-s)^{5/2} \tag{19}$$

3. Construction of Partial Wave Amplitudes

Following the work by Chew & Mandelstam (1960) and Brown & Glol (1968) we represent the partial waves in the following way

$$f_l^I(s) = \frac{k^{2l}}{(H_l^I(s)+k^{2L}h(s))+1/N_l^I(s} \tag{19}$$

where $H_1^I(s)+k^{2l}h(s))$ forms the conventional D function In this formulation,

i) $N_l^I(s)$ is a real analytic function with the left-hand cut.

ii) $H_l^I(s)$ is a mermorphic function except for a right hand inelastic cut from $s = 4$ ro $s = \infty$.

iii) $h(s)$ is a real analytic function whose only singularity is a cut from $s = 1$ to $s = \infty$, with discontinuity $-k/(2\sqrt{s})$

To construct $H_l^I(s)$ and $N_l^I(s)$ we take the help of the optimized polynomial technique due to Cutkosky & Deo (1968) The technique is to select, from functions of a given class, a rapidly convergent sequence of functions for use in fitting experimental data This sequence is derived by suitable conforal mapping The advantage of this has been shown by Frazer (1961) for the effective range theory There he has demonstrated by numerical example that a fit by a polynomial of the fourth order in the original variable is as good as a fit by a polynomial of the second order in the new mapped variable There has also been a large number of successful applications (Deo & Parida 1971, 1973, 1974) of the proposed method.

The crossing constraints in the form of Roskies' relations and Martin's inequalities have been given for the unphysical region $0 \leqslant s \leqslant 1$, where as all the available experimental data are in the physical region $s > 1$. To calculate the integrals of Roskies' relations, the values of the partial wave amplitudes are to be known from $s = 0$ to $s = 1$ Hence for our analysis we will treat the region $0 \leqslant s \leqslant 1$ as the *physical domain* for the analytic extrapolation

As a first step to achieve this, we make the simple square root mapping

$$W = \sqrt{s}. \quad \dots \tag{20}$$

This symmetrizes separately the left hand and right hand inelastic cuts, respectively on the imaginary and real axes of the complex W plane, on both sheets (figure 1(b)).

Now the optimal convergence can be obtained by mapping the symmetrical cuts of the W plane to form the boundary of a unifocal ellipse with $W = \pm 1$ as foci. But the size of the ellipse is quite large (semi-major axis $= 3.790$, semi-minor axis $= 3.662$) So for simplicity the circular mappings were preferred In the sense that no region of analyticity is left out of the region of convergence by this mapping, the expansion of the partial wave amplitudes in the mapped variable will converge optimally

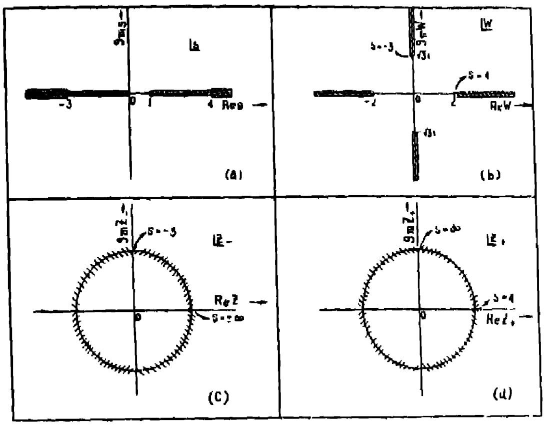

Fig 1 · (a) The analytic structure of the partial wave amplitudes in the complex s-splane, showing the elastic and inelastic cuts

(b) The structure of the inelastic cuts of the complex s-plane in the complex W-plane.

(c) The cuts on the imaginary axis of the W plane mapped on to the of a circle of radius $(2+\sqrt{3})$ in the Z_- plane

(d) The cuts on the real axis of the W plane mapped on to the boundary of a circle of radius $(2+\sqrt{3})$ in the Z_+ plane

The cuts, arising out of the left hand inelastic cut of the complex s-plane, on the imaginary axis of W-plane are mapped on to the boundary of a circle of radius $(2+\sqrt{3})$, by the mapping (figure 1(c)).

$$Z_- = (\sqrt{3+W^2}-\sqrt{3})(2+\sqrt{3})/W \tag{21}$$

Next the cuts on the real axis of the W-plane, which are due to the right hand inelastic cut on the complex s-plane, are mapped on to the circumference of a circle of the same radius, $(2+\sqrt{3})$, by the mapping (figure 1(d)).

$$Z_+ = (2+i\sqrt{W^2-4})(2+\sqrt{3})/W \quad \ldots \tag{22}$$

In both the cases the radii, $(2+\sqrt{3})$, of the circles are so chosen that the unphysical region $(0,1)$ of the s-plane is now mapped on to the interval $(0, 1)$ of the mapped plane For large s, the real part of the mapped variable Z_+ vanishes as $1/\sqrt{s}$ where as the imaginary part tends to the constant $(2+\sqrt{3})$

The functions, H_l^I and N_l^I can be constructed as expansions in polynomials of Z_+, and Z_- respectively

For I = 0 and 1 :

H_l^I of the D-function is written as

$$H_l^I = \sum_{}^{\infty} b^I_{l,n}\, Z_+^n + \delta_{l,1} s \sum_{}^{\infty} C_{l,\,n}\, Z_+^n + \frac{\lambda}{s-s_0}\, \delta_{l,\,0}\delta_{I,\,0}. \tag{23}$$

The second term is a subtraction for P-wave and only $n = 0$ term of this series is enough. This is so because the partial waves fall off as s^{-1}, (Eden 1967), and the S-wave presumably goes to a constant. The third is a CDD pole for the I = 0, S-wave with position s_0 and strength λ of the pole as parameters to be determined. We have searched for values of s_0 close to the energy where the phaseshift goes to 180° *i.e.* around 990 MeV

The equivalent N-function, N_l^I, can be parametrised as

$$N_l^I = \sum_{}^{\infty} C_{l,n}^{I'} Z_-^n (1-\delta_{n,\,1}). \tag{24}$$

Due to the analytic constraints, eqs (16), (17) and (18) one has to ensure the absence of any term containing Z . It is found convenient to determine the coefficients by writing the N's in the following way

$$N_l^{I'} = C_l^{I'}{}_{,0} + C_l^{I'}{}_{,2} Z_-^2 + C_l^{I'}{}_{,3} Z_-^2(1-Z_-) + \ldots \tag{25}$$

Additions of more terms did not improve the χ^2.

To construct $h(s)$ we follow the usual dispersion relation technique and write,

$$h(s) = \frac{1}{\pi}\int_0^{\infty} \frac{Im h(s')}{s'-s}\, ds' \tag{26}$$

$$= \frac{1}{\pi}\int_1^{\infty} \frac{\sqrt{s'-1}/(2\sqrt{s'})}{s'-s}\, ds' \tag{27}$$

The integration is performed directly and one obtains,

$$h(s) = \frac{k}{2\sqrt{s}}\left[\frac{2}{\pi}\log(\sqrt{s}+k) - i\right]. \tag{28}$$

The function $h(s)$ thus constructed is real in the unphysical region $0 \leqslant s \leqslant 1$, has a discontinuity $-k/(2\sqrt{s})$ for $s > 1$ and does not possess the spurious $\sqrt{s}$ type singularity near $s = 0$. Furthermore, this function removes the elastic cut from the D-function from all sheets. However for large s the real part of $k^{2l}h(s)$ increases logarithmically whereas the imaginary part tends to a constant for the S-wave. For the P-wave, the real part goes as $s \log(s)$ with the imaginary part increasing as s We expect, however, the imaginary part to predominate at higher energies. Roos & Pisut (1969) have shown that the precise form of the parametrization of the D function is not important so long as the P-wave kinematical factor is present in $Im\,D$. Hence we need modify $h(s)$ in such a way that as $s\to\infty$ the real part of $h(s)$ tends at best to a constant. This we achieve by writing the modified $h(s)$ as,

$$h(s) = h(s)+h_1(s), \quad \ldots \quad (29)$$

where

$$h_1(s) = \frac{1}{\pi} \log \left(\left| 1 + \frac{Z_1^2}{(2+\sqrt{3})^2} \right| \right) \quad \ldots \quad (30)$$

The functions H_l^I and N_l^I have been constructed using the ideas of optimized polynomial expansion. It is desirable to have definite values for the degree of accuracy obtained by truncating the expansion. This is achieved by the use of a convergence test function (CTF) (Miller *et al* (1972) Cutkosky (1972), Chao (1970)) which is added to the usual χ_0^2 used in fitting theoretical formula to given data points. This CTF is based on the ansatz that the parameters can be treated as independent Gaussian random variables with a variance depending only on the radius of convergence (R) and their order of occurrence in the polynomial expansion. For an expansion like ours it is found convenient to follow Cutkosky (1972) who has suggested the following expression for the CTF,

$$\phi = \sum_{N=1}^{N} P_N, \quad \ldots \quad (31)$$

$$P_N = N \log \left(1+ \frac{a_N^2}{S_N} \left(1-\frac{0.58}{N}\right)\right), \quad \ldots \quad (32)$$

$$a_N^2 = C_N^2 b_N^2, \quad \ldots \quad (33)$$

$$S_N = \sum_{N=0}^{N-1} a_N^2, \quad \ldots \quad (34)$$

and

$$C_N^2 = \frac{1}{3}(2\delta_{N,0}+R^{2N}) \quad \ldots \quad (35)$$

In our case $R = 2+\sqrt{3}$

The total χ^2 is now $\chi^2 = \chi_0^2+\phi$.

Some features of the CTF of eq. (31) are important to mention here. For large R, P_N depends logarithmically on R_0. If at some point of the expansion the a_N has got a magnitude close to the noise level of the data (in our case Roskies' relations and Martin's inequalities) the inclusion of ϕ provides a constraint which makes the a_N's a diminishing sequence, for otherwise the χ^2 will be large. However, if the data really require some a_N to be large, the logarithmic nature of P_N ensures that the constraint is partially turned off. Thus one can take a conservatively large number of parameters without sacrificing the uniqueness and stability of the fit. This means that even though one takes a large number of parameters in the expansion still the number of degrees of freedom remains the same.

4. Details of Calculations

After some preliminary searches it was soon realised that three terms in each N_l^I, written as in eq (25), were enough to satisfy the Martin's inequalities and Roskies' integral relations to a good degree of accuracy. Two parameters correspond essentially to finding position and strength of the zeros of the partial wave amplitudes in the region $0 \leqslant s \leqslant 1$ and the third is fixed by the relations (16), (17) and (18) near $s = 0$

Since inelasticities for $s > 16m_\pi^2$, however small, are bound to be present at least two terms in each H_l^I's have to be taken We find that the inelastic coefficients exceed one as we calculate them at higher and higher energies and more terms become necessary in H_l^I's In case of $l = 0$, $I = 0$ wave, the inelastic coefficient could be kept below 1, only by introducing a CDD pole

The progressive results are tabulated. In tables 1, 2 and 3 results of the search for only the two S-waves related by the Roskies' first two relations are given In table 4 the result of search for the P-wave, related to the S-wave through the Roskies' last three relations are given. For filling each row, a large number of searches with fairly arbitrary starting points were made Only the final results are given in the tables In table 3, results by introducing a CDD pole in $l = 0$, $I = 2$ wave is also given for comparison purposes Taking partial waves with more than seven parameters did not improve the results any more

5 Numerical Results

From tables 1, 2, 3 and 4 it is obvious that to keep the inelasticities less than unity at least seven parameters per wave is necessary and as such we report the results obtained from it With eight parameters per wave the results did not improve.

Since most of our inputs are in the unphysical region and then we extrapolate our prescription to the physical region, we obviously divide our results into two parts

(a). *Results in the region* $0 \leqslant s \leqslant 1$.

(i) The values of the two sides of Roskies' sum rules, which were our input, are given in table 5. The first four sum rules are satisfied within an error of 0 5 percent The fifth sum rule, though appears to be violated, is still valid to a much better extent than those obtained from various models (Krinsky 1970 Bonnier & Gauron 1970) This shows that our analysis is a definite improvement over those obtained so far,

Table 1. Progressive results of the search programme with a *CDD* pole in $I = 0$, S-wave.

No. of parameters: $I = 0$, S-wave H_l^I	N_l^I	$I = 2$, S-wave H_l^I	N_l^I	Discrepancy in Roskies' first two relations	No. of Martins inequalities satisfied	No. of Penningtons' inequalities satisfied	Remarks on the extrapolated results: η_0^0	η_0^2	δ_0^0	δ_0^2
5	2+1	5	2+1	<0.5%	All	All	<1.0	<1.0	fits well	fits well
4	2+1	4	2+1	<0.5%	All	All	$\leqslant$1.0 for 560 MeV < c.m. energy < 650 MeV	>1.0	shows a hump at c.m. energy of 750 MeV	does not fit for c.m. energy 1300 MeV
3	2+1	3	2+1	<0.5%	All	All	$\leqslant$1.0 for c.m. energy 1050 GeV	>1.0	show a humps at c.m. energy of 690 MeV	does not fit for c.m. energy 1200 MeV
2	2+1	2	2+1	<0.5%	All	All	<1.0	>1.0	does not at at low energies	does not fit
1	2+1	1	2+1	<0.5%	All	All	= 1.0	= 1.0	does not fit	does not fit
0	2+1	0	2+1	<0.5%	4	9	−1.0	−1.0	does not fit	does not fit

Table 2. Progressive results of the search programme with no *CDD* pole in either $I = 0$, S-wave or $I = 2$, S-wave.

No of parameters				Discrepancy in Roskies first two relations	No of Martins' inequalities satisfied	No. of Penningtons inequalities satisfied	Remarks on the extrapolated results			
$I = 0$, S-wave		$I = 2$, S-wave								
H_1^I	N_1^I	H_1^I	N_1^I				η_0^0	η_0^2	δ_0^0	δ_0^2
5	2+1	5	2+1	<0.5%	All	All	>1	>1	does not fit for c.m. energy 900 MeV	fits well
4	2+1	4	2+1	<0.5%	All	All	>1	>1	-do-	does not fit for c.m. energy 1300 MeV
3	2+1	3	2+1	<0.5%	All	All	>1	>1	-do-	does not fit for c.m. energy 1200 MeV

Table 3. Progressive results of the search programme with a *CDD* pole in each of the $I = 0$ and $I = 2$, S-wave.

No. of parameters				Discrepancy in Roskies' first two relations	No. of Martins' inequalities satisfied	No. of Penningtons' inequalities satisfied	Remarks on the extrapolated results			
$I = 0$, S-wave		$I = 2$, S-wave								
H_1^I	N_1^I	H_1^I	N_1^I				η_0^0	η_0^2	δ_0^0	δ_0^2
5	2+1	5	2+1	<0.5%	All	All	<1.0	<1.0	fits well	fits well
4	2+1	4	2+1	<0.5%	All	~~All~~	>1.0 for 500 MeV c.m. energy ⩽650 GeV	>1.0	shows a hump at c.m. energy 750 MeV	does not fit at high energies
3	2+1	3	2+1	<0.5%	All	All	>1.0 for c.m. energy 1050 MeV	>1.0	shows a hump at c.m. energy 690 MeV	does not fit at high energies

Table 4. Progressive results of the search programme of the *P*-wave.

No. of parameters		Disc in the 3rd and 4th relations	in the 5th Roskies' relations	[illegible] extrapolated results	
H_1^I	N_1^I			η_1^1	δ_1^1
5	2+1	$<0.5\%$	33%	≈ 1 but less than unity	fits well
4	2+1	$<0.5\%$	30%	< 1.0 for c.m. energy 1100 MeV	fits well
3	2+1	$<0.5\%$	30%	>1.0	fits poorly
2	2+1	$<0.5\%$	23%	$\simeq 1.0$	does not fit
1	2+1	$<0.5\%$	15%	$\simeq 1.0$	does not fit
0	2+1	$<0.5\%$	13%	$\simeq 1.0$	does not fit

Table 5. Numerical values of Roskies' sum rules.

Roskies' Sum rule	L.H.S.	
R-1	0.2302×10^{-1}	0.2300×10^{-1}
R-2	0.1849×10^{-1}	0.1853×10
R-3	0.2778×10^{-1}	0.2786×10^{-1}
R-4	0.6079×10^{-2}	0.6087×10^{-2}
R-5	0.4432×10^{-3}	$0.3415 \times 10^{-}$[illegible]

(ii) The partial wave amplitudes constructed by us have the following characteristics.

(1) $f_0{}^0$ has a zero at $s = 0.1218$

(2) $f_0{}^2$ has a zero at $s \doteq 0.468$

(3) $f_1{}^1$ has the trivial zero at $s = 1$.

The results are in good agreement with those predicted from current algebra models (Krinsky 1970, Kang & Lee 1971). The behaviour of our partial wave amplitudes in the unphysical region is shown in figure 2.

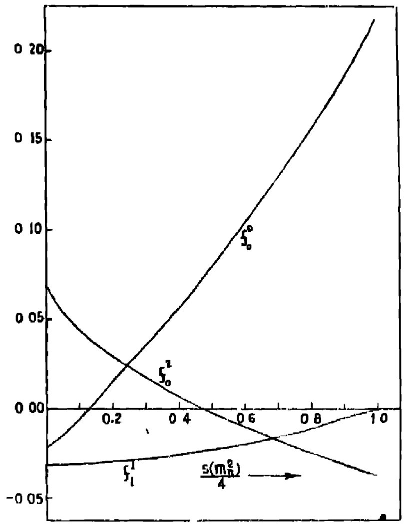

Fig. 2 : Behaviour of partial wave amplitudes $f_0{}^0$, $f_0{}^2$ and $f_1{}^1$ in the unphysical region $0 \leqslant s \leqslant 1$

(iii) The S-wave neutral $\pi-\pi$ amplitude defined by eq (10) is plotted in figure 3 The neutral $\pi-\pi$ amplitude due to Krinsky (1970), (Kr), Kang & Lee (1971), (KL), and Bonnier & Gauron (1972), (BG), are also plotted in the same figure The Kr curve shows a minimum at $s = 0.41$, where as KL curve and BG show minimum at $s = 0\ 407$ and $s = 0.400$ respectively As compared with this curve our curve shows a minimum at $s = 0\ 346$ and $f_0(1) > f_0(s)$ for $0 \leqslant s < 1$. Thus Martin's inequalities (M-4), (M-5) and M-6) are automatically satisfied. The values of the other three inequalities are given in table 6.

(iv) The two inequalities refered to in Kang's (1971) work are satisfied in the following way.

$$f_0\,(0.07342) < f_0\,(0.60565) \qquad \dots\ (36)$$

$$0.03047 < \quad 0.02723$$

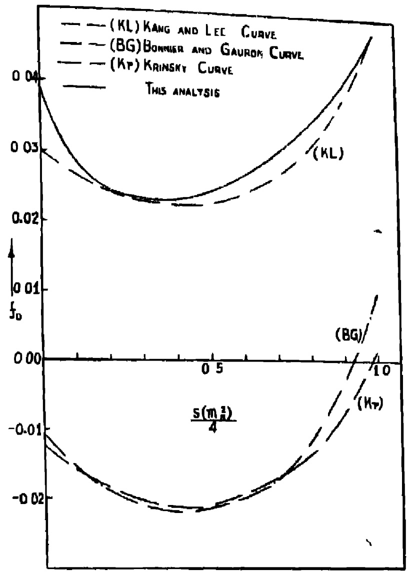

Fig. 3 : Behaviour of the neutral amplitude f_0 as obtained from this analysis in the unphysical region $0 \leqslant s \leqslant 1$ is drawn in bold line KL is Kang and Lee's, BG is Bonnier and Gauron's and Kr is Krinsky's curve for the neutral wave in the same region.

$$f_0\,(0\,84717) > f_0\,(0\,12007) \qquad . \qquad (37)$$
$$0\,03688 \;>\; 0.02778$$

(v) The L.H S of Pennington's (1970, 1971) inequalities, as obtained from our analysis, are given in tables 7 and 8

(vi) Defining the scattering length as

$$a_l^I = \lim_{s \to t} - \left[\frac{k^{2l+1}}{2\sqrt{s}} \cdot \cot \delta_l^I \right]^{-1} \qquad . \qquad (38)$$

We obtain from our analysis

$$a_0^0 = -0\,2188$$
$$a_0^2 - 0\,03801$$
$$a_1^1 = -0\,054$$

Table 6. Numerical values of Martin's inequalities, *M*-1, *M*-2, and *M*-3.

Martin's inequality	L H.S.	R H.S
M-1	0.03950	0 03397
M-2	0.03460	0 03199
M-3	0 03229	0.03218

so that the ratio $-a_0^0/a_0^2 = 5.75$ As compared with this Weinberg (1966) had obtained $a_0^0 = -0.2$ and $a_0^2 = 0\ 06$, with the ratio $-a_0^0/a_0^2 = 3\ 3$ Recently Brandt *et al* (1972) replaced the assumptions of strong pion pole dominance of Weinberg by a much weaker pion pole dominance. Then using the light cone and Regge pole techniques they have obtained by scattering lengths as $a_0^0 = -0\ 24$, $a_0^2 = 0.07$ On the other hand results of Kang & Lee (1971) are $a_0^0 = -0\ 219$, $a_0^2 = 0.039$, and $-a_0^0/a_0^2 = 5.6$. More recently Morsan (1972) has predicted the values $a_0^0 = -0.21$, $a_0^2 = 0.037$, and $-a_0^0/a_0^2 = 5.7$ Thus our S-wave scattering lengths are in good agreement with those obtained from various models

The P-wave scattering length obtained by Weinberg is $a_1^1 = -0\ 058$. Olsson's (1967) value is -0.061 Brehm *et al* (1969) have obtained -0.067 and Moregan's (1972) result is $-0\ 036$. Thus our result is of the same order

(b) *Results in the region $s > 1$, on extrapolation*

(i) The $I = 0$, S-wave phase shifts are plotted in figure 4 It is in good agreement with the experimental data of Protopopescu *et al* (1973) and Flatte *et al* (1972) The phase shift below 700 MeV continues to rise slowly. Between 700 to 900 MeV the phase shifts remain in the range $80° \pm 5°$, showing a broad resonance Beyond 900 MeV the phase shift shoots up sharply and since in our search programme we obtained a value of s_0 which corresponds to an approximate energy value of 991 MeV, the phase shift crosses 180° at the point For energies more than 1020 MeV the phase shift remains almost flat at the value 200°.

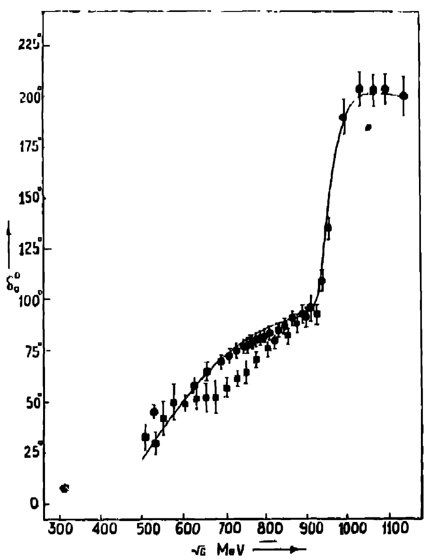

Fig. 4. The δ_0^2phase shifts upto 1150 MeV, data points of Boton *et al* (1970), data p[oints] of Protopopscu *et al* (1973)

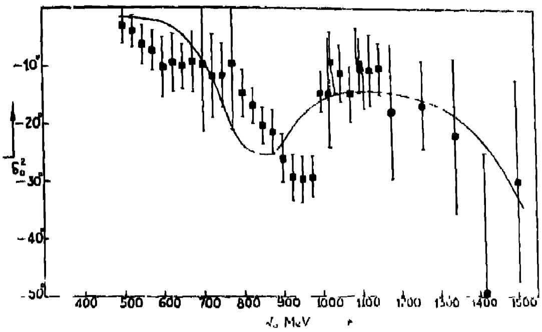

Fig 5 : The δ_0^2 phase shifts upto 1400 MeV., ■ data points of Boton *et al* (1970, ● data points of Carroll *et al* (1972).

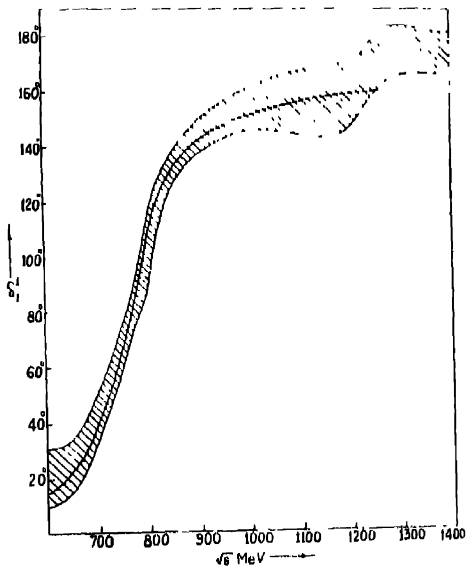

Fig 6 : The δ_1^1 phase shifts upto 1400 MeV, |\\\\\\| data band of Carroll *et al* (1972)

However, we also observed that a set of parameters which made the partial wave amplitudes somewhat satisfy the crossing conditions and gave phase shifts analogous to either the *down-down* results of Oh *et al* (1970) and Caroll *et al* (1972) or the *up-up* results of Baton *et al* (1973) made the inelasticity η_0^0 more than unity, which is unphysical. Thus by our technique we have succeeded in eliminating, at least qualitatively, the *up-up* solution over the whole region and the *down-down* solution in the region greater than 900 MeV

Table 7. Numerical values of the left hand side of Pennington's inequalities from to P-1 P-2.

P-1	P-2	P-3	P-4	P-5	P-6	P-7
$0.25602\times10^{+3}$	$0.29252\times10^{+3}$	0.11952×10^{-3}	0.78949×10^{-4}	0.30217×10^{-4}	0.35497×10^{-4}	0.24937×10^{-4}

Table 8. Numerical values of the left hand side of Pennington's inequalities from P-8 to P-13.

ϵ	P-8	P-9	P-10	P-11	P-12	P-13
0.0	0.40578×10^{-4}	0.12052×10^{-2}	$0.11268\times10^{+3}$	0.90207×10^{-4}	0.31357×10^{-3}	0.28205×10^{-3}
0.1	0.21793×10^{-4}	0.82462×10^{-3}	0.53363×10^{-4}	0.55604×10^{-4}	0.24018×10^{-3}	0.20908×10^{-3}
0.2	0.13249×10^{-4}	0.57548×10^{-3}	0.33292×10^{-4}	0.32955×10^{-4}	0.16604×10^{-3}	0.14806×10^{-3}
0.3	0.14946×10^{-4}	0.45781×10^{-3}	0.52473×10^{-4}	0.22258×10^{-4}	0.12414×10^{-3}	0.98999×10^{-4}
0.4	0.26884×10^{-4}	0.47163×10^{-3}	0.11090×10^{-3}	0.23514×10^{-4}	0.81482×10^{-4}	0.61884×10^{-4}
0.5	0.49063×10^{-4}	0.61693×10^{-3}	0.20859×10^{-3}	0.36723×10^{-4}	0.49063×10^{-4}	0.36723×10^{-4}
0.6	0.81482×10^{-4}	0.89370×10^{-3}	0.34552×10^{-3}	0.61885×10^{-4}	0.26884×10^{-4}	0.23514×10^{-4}
0.7	0.12414×10^{-3}	0.13019×10^{-2}	0.52171×10^{-4}	$\cancel{0.14946}\times10^{-4}$	0.14946×10^{-4}	0.22258×10^{-8}
0.8	0.17704×10^{-3}	0.18417×10^{-2}	0.73715×10^{-3}	0.14768×10^{-3}	0.13249×10^{-4}	0.32954×10^{-4}
0.9	0.24108×10^{-3}	0.25129×10^{-2}	0.99185×10^{-3}	0.20908×10^{-3}	0.21794×10^{-4}	0.55604×10^{-4}
1.0	0.31357×10^{-3}	0.33156×10^{-2}	0.12857×10^{-2}	0.28205×10^{-3}	0.40579×10^{-4}	0.90206×10^{-4}

Through the search programme we obtained a value of 0 95 for λ.

(ii) The $I = 2$, S-wave phase shifts are plotted in figure 5. They are repulsive and agree well with the experimental data up to 1450 MeV.

(iii) The $I = 1$, P-wave phase shifts, plotted in figure 6 shows a resonance at 767.7 MeV and fits well with the experimental data up to 1300 MeV and beyond it agrees only qualitatively with the same

(iv) We have obtained a value of 30° for $(\delta_0^0(m_k)-\delta_0^2(m_k))$ Krinsky's (1970) value for the same is 44° The experimental value (Gutay *et al* (1969)) for the same is $50° \pm 20°$.

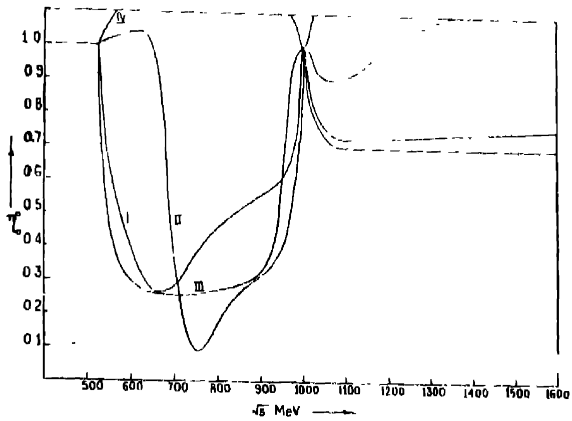

Fig. 7 : The inelasticity η_0^0 upto 1600 MeV, with I-5 parameters, II-4 parameters, III-3 parameters and IV-2 parameters in the H_0^0 function.

(v) The $I = 0$, S-wave inelasticity is plotted in figure 7. The different curves corresponding to various number of parameters in H_l^I pass through the the point $\eta_0^0 = 1$ at approximately 1000 MeV because at that cm energy the real part of $|f_0^0|^{-1}$ tends to ∞ The value for η_0^2 and η_1^1 are also plotted in figure 8 and figure 9 respectively. They remain close to unity

(vi) f_0^0 has a zero at $s = 12\ 5$.

6. Conclusion

In conclusion we repeat that analyticity, crossing symmetry and unitarity puts severe constaints on the $\pi-\pi$ scattering amplitude in the unphysical region. However, the questions that have been bothering workers are that, can this knowledge of $f_l(s)$ in the unphysical region lead to an absolute determination of the function $f_l(s)$ in the elastic and inelastic regions ? Or are they marginal conditions which can always be accomodated ? Or for that matter are some types of parametrization excluded ? Wanders & Piguet (1968) have shown that the

Table 9. Values of parameters.

$n =$	0	1	2	3	4
b_0^0, n	−3.3377	2.0862	−0.91935	0.16659	−0.01198
b_0^2, n	−7.1874	6.2382	−3.9722	0.84653	−0.01191
b_1^1, n	17.009	−2.1239	0.55767	−0.06169	
c_1^1, n	−2.4322				
$c_0^{0\prime}, n$	−0.023		0.17346	−0.01123	
$c_0^{2\prime}, n$	0.04726		−0.09244	−0.003938	
$c_1^{1\prime}, n$	0.06573		0.12032	−0.03459	

problem of constructing models of the s-wave $\pi-\pi$ scattering verifying all known exact properties is not trivial, even in the neutral theory. This means that the physical content of these conditions is not empty. In a continuation (Wanders & Piguet 1972) of this work, they have imposed quasilinearity on the amplitudes in the unphysical region. Their analysis shows that the rigorous constraints below threshold have a physical relevance which is restricted to a small energy interval above threshold. In our analysis, without taking help of the results from various models we have used these constraints coupled with the ideas of optimized polynomial expansion and the CTF and have constructed partial wave amplitudes which describes the $\pi-\pi$ scattering not only in the low energy limits but also in the inelastic region upto 1.4 GeV. A CDD pole in S-wave at the value of $s = 12.5$ is found to be very much necessary. However, tables of the progressive results of the search program and figures 7, 8 and 9 clearly indicate that over and above the constraints in the unphysical region one has to fruitfully use the effect of the inelasticities so as to determine the minimum number of parameters per wave. The large inelasticities near 850 MeV $l = 0$, $I = 2$ and near 750 MeV for $l = 1$, $I = 1$ are due to opening out of the inelastic channels at $16m_\pi^2$. We have not searched for the Odorico (1972) zeroes. The amplitudes, constructed by us, may not be unique but we assert that they are a possible best.

A possible line of improved calculations is to use Ray's (1971) equations. Here, however, one has to parametrise many partial waves. A phenomenological analysis of $\pi\pi$ scattering using the above conclusion has since then been carried out by us (Deo & Mohapatra 1975). For this we used the strip mapping This not only reduces the number of parameters to only five for $I = 0$ S-wave, but also gives a good fit to the experimental curve for η_0^0 which could not be predicted accurately in the present work.

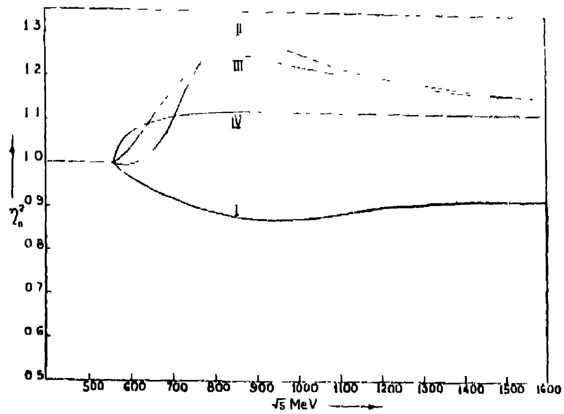

Fig. 8 : The inelasticity η_0^2 upto 1600 MeV, with I-5 parameters, II-4 parameters, III-3 parameters and IV-2 parameters in the H_0^2 function

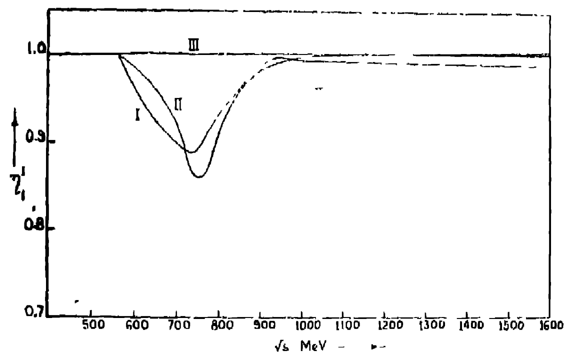

Fig. 9 . The inelasticity η_1^1 upto 1600 MeV, with I-5 parameters, II-4 parameters and III-3 parameters in the H_1^1 function.

The calculations have been carried out in the IBM 1130 Computer of the Computer Centre of the Utkal University We are thankful to the staff for their help and cooperation. We are also thankful to Dr. R. C. Das for encouragements.

Appendix

The Pennington's inequalities are,

$$\int_0^1 ds(1-s)(2s^2-s)f_0(s) \geqslant 0 \qquad \ldots \quad (P\text{-}1)$$

$$\int_0^1 ds(1-s)(5s^3-3s^2)f_0(s) \geqslant 0 \qquad \ldots \quad (P\text{-}2)$$

$$\int_0^1 ds(1-s)(-5s^3+7s^2-2s)f_0(s) \geqslant 0 \qquad \ldots \quad (P\text{-}3)$$

$$\int_0^1 ds(1-s)(-s^4-s^3+3s^2-s)f_0(s) \geqslant 0 \qquad \ldots \quad (P\text{-}4)$$

$$\int_0^1 ds(1-s)(18s^4-32s^3+18s^2-3s)f_0(s) \geqslant 0 \qquad \ldots \quad (P\text{-}5)$$

$$\int_0^1 ds(1-s)(21s^5-15s^4-20s^3+20s^2-4s)f_0(s) \geqslant 0 \qquad \ldots \quad (P\text{-}6)$$

$$\int_0^1 ds(1-s)(-21s^5+51s^4-44s^3+16s^2-2s)f_0(s) \geqslant 0 \qquad \ldots \quad (P\text{-}7)$$

$$\int_0^1 ds(1-s)[(s^4-4s^3+4s^2-s)-2\epsilon(-5s^3+7s^2-2s)$$
$$+2\epsilon^2(2s^2-s)]f_0(s) \geqslant 0, \quad 0 \leqslant \epsilon \leqslant 1 \qquad \ldots \quad (P\text{-}8)$$

$$\int_0^1 ds(1-s)[(49s^5+130s^4-395s^3+285s^2-57s)-110\epsilon(s^4-4s^3+4s^2-s)$$
$$+55\epsilon^2(-5s^3+7s^2-2s)]f_0(s) \geqslant 0 \quad , \quad 0 \leqslant \epsilon \leqslant 1 \qquad \ldots \quad (P\text{-}9)$$

$$\int_0^1 ds(1-s)[(7s^5-15s^4+5s^3+5s^2-2s)-10\epsilon(-s^4-s^3+3s^2-s)$$
$$+5\epsilon^2(5s^3-3s^2)]f_0(s) \geqslant 0; \quad 0 \leqslant \epsilon \leqslant 1 \qquad \ldots \quad (P\text{-}10)$$

$$\int_0^1 ds(1-s)[(-7s^5+20s^4-25s^3+15s^2-3s)-10\epsilon(s^4-4s^3+4s^2-s)$$
$$+5\epsilon^2(-5s^3+7s^2-2s)]f_0(s) \geqslant 0; \quad 0 \leqslant \epsilon \leqslant 1 \qquad \ldots \quad (P\text{-}11)$$

$$\int_0^1 ds(1-s)[(s^4+6s^3-6e^2+s)-2\epsilon(5s^3-3s^2)$$
$$+2\epsilon^2(2s^2-s)]f_0(s) \geqslant 0\,; \quad 0 \leqslant \epsilon \leqslant 1 \qquad \ldots \quad (P\text{-}12)$$

$$\int_0^1 ds(1-s)[(-7s^5+10s^4-10s^3+10s^2-3s)-10\epsilon(-s^4-s^3+3s^2-s)$$
$$+5\epsilon^2(-5s^3+7s^2-2s)]f_0(s) \geqslant 0; \quad 0 \leqslant \epsilon \leqslant 1 \qquad \ldots \quad (P\text{-}13)$$

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